Point charge in presence of grounded Conducting sphere:-

Let us consider a sphere centred at O. and radius is a.

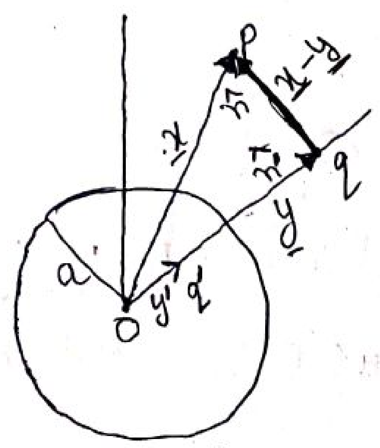

fig. Conducting sphere Of radius a centred about origin.

q charge placed at at y.

and observation point P at a distance x. and q is placed at position y. y' is position of induced charge q'. q is outside from sphere then q' is inside. n and n' denotes the direction of x and y.

Consider prob. of fig (a) in which q is point charge located at y related to origin around which grounded conducting sphere of radius a. we, see potential $\phi(x)$ s.t.

$$\phi(x = a) = 0$$

, by symmetry it is evedent that image charge q' will lie on the ray from origin to charge q. If we consider q to the outside of the sphere image position y' will be inside the sphere. Now, the potential due to charge q and q' at a point P, at a distance

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\underline{x}-\underline{y}|} + \frac{q'}{|\underline{x}-\underline{y}'|}\right] \longrightarrow (1)$$

we will calculate the potential at a (surface of sphere), ~~then~~

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|x\hat{n}-y\hat{n}'|} + \frac{q'}{|x\hat{n}-y'\hat{n}'|}\right] \longrightarrow (2)$$

Now, we have to try the value of q', y' s.t. $\phi(x=a)$ vanishes at surface of the sphere.

magnitude location

~~Now~~, If x is factored out from first term and y' from (IInd) at $|x|=a$,

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{x\left|\hat{n}-\frac{y}{x}\hat{n}'\right|} + \frac{q'}{y'\left|\frac{x}{y'}\hat{n}-\hat{n}'\right|}\right]$$

$$\phi(x=a) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{a\left|\hat{n}-\frac{y}{a}\hat{n}'\right|} + \frac{q'}{y'\left|\frac{a}{y'}\hat{n}-\hat{n}'\right|}\right] \longrightarrow (3)$$

Now, from (3) It is clear that if $\frac{q}{a} = -\frac{q'}{y'}$ — (a)

and, $\frac{y}{a} = \frac{a}{y'}$ — (b)

will make $\phi(x=a)=0$ for all possible values of $\hat{n}\cdot\hat{n}'$.

from (B) location

$$y' = \frac{a^2}{y}$$

and, magnitude

$$q' = -\frac{a}{y}\cdot q$$

$\longrightarrow$ (*)

magnitude & location given by (4). We note that as q is brought closer to sphere image charge grows in magnitude and moves out from centre of sphere. When q is just outside at the surface of sphere image charges are equal and opposite in magnitude ~~and y approaches a~~ and lies just beneath surface

~~Induced surface charge density at the surface~~

$$\cancel{\sigma = -\varepsilon_0 \left.\frac{\partial \phi}{\partial x}\right|_{x=a}}$$

As image charge has been found we can return to the original prob of charge q on conducting grounded sphere, actual surface charge density induced on the surface of the sphere can be calculated normal derivative of $\phi(x)$ at surface i.e

$$\sigma = -\varepsilon_0 \left.\frac{\partial \phi}{\partial x}\right|_{x=a}$$

$$\phi(x) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right] \qquad (\hat{n}\cdot\hat{n}' = \cos\gamma')$$

$$= \left[\frac{q}{\sqrt{x^2 \cdot n^2 - y^2 \cdot n'^2 + 2xy \cdot n \cdot n'}} + \frac{q'}{\sqrt{x^2 n^2 - y'^2 \cdot n'^2 + 2xy' \cdot n \cdot n'}}\right]$$

$$\left(\because n^2 = n'^2 = 1, \quad n \cdot n' = \cos\gamma\right)$$

$$\phi(x) = \left[\frac{q}{\sqrt{x^2 + y^2 - 2xy\cos\gamma}} + \frac{q'}{\sqrt{x^2 + y'^2 - 2xy'\cos\gamma}}\right]$$

$$\frac{\partial \phi}{\partial x} = -\frac{1}{2}\, q\, \frac{(2x - 2y\cos\gamma)}{(x^2 + y^2 - 2xy\cos\gamma)^{3/2}} - \frac{1}{2}\, \frac{q'(2x - 2y'\cos\gamma)}{(x^2 + y'^2 - 2xy'\cos\gamma)^{3/2}}$$

$$= \frac{-q(x-y\cos\gamma)}{(x^2+y^2-2xy\cos\gamma)^{3/2}} - q'\frac{(x-y'\cos\gamma)}{(x^2+y'^2-2xy'\cos\gamma)^{3/2}}$$

$$\frac{\partial\phi(x)}{\partial x} = -\left[\frac{q(x-y\cos\gamma)}{(x^2+y^2-2xy\cos\gamma)^{3/2}} + \frac{q'(x-y'\cos\gamma)}{(x^2+y'^2-2xy'\cos\gamma)^{3/2}}\right]$$

$\cancel{q'z} = -\frac{aq}{y}$,

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=a} = -\frac{1}{4\pi\varepsilon_0}\left[\frac{q(a-y\cos\gamma)}{(a^2+y^2-2ay\cos\gamma)^{3/2}} + \frac{q'(a-y'\cos\gamma)}{(a^2+y'^2-2ay'\cos\gamma)^{3/2}}\right]$$

but, $q' = -\frac{aq}{y}$, $y' = \frac{a^2}{y}$

then,

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=a} = -\frac{1}{4\pi\varepsilon_0}\left[\frac{q\cdot(a-y\cos\gamma)}{(a^2+y^2-2ay\cos\gamma)^{3/2}} - \frac{aq\left(a-\frac{a^2}{y}\cos\gamma\right)}{y\,(a^2+y'^2-2ay'\cos\gamma)^{3/2}}\right]$$

$$= -\frac{1q}{4\pi\varepsilon_0}\left[\frac{(a-y\cos\gamma)}{y^3\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}} - \frac{a\cdot a\left(1-\frac{a}{y}\cos\gamma\right)}{y\left(a^2+\frac{a^4}{y^2}-2a\cdot\frac{a^2}{y}\cos\gamma\right)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{a(a-y\cos\gamma)}{a\cdot y^3\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}} - \frac{y\cdot \cancel{a^2}(y-a\cos\gamma)}{yy^2\cdot a^{\cancel{3}}\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{a^2-ay\cos\gamma}{a\,y^3\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}} - \frac{(y^2-ay\cos\gamma)}{y^3\cdot a\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{a^2-\cancel{ay\cos\gamma}-y^2+\cancel{ay\cos\gamma}}{y^3\cdot a\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}\right]$$

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$$\left.\frac{\partial \phi}{\partial x}\right|_{x=a} = -\frac{q}{4\pi\varepsilon_0} \cdot \frac{(a^2-y^2)}{y^3 \cdot a\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$= +\frac{q}{4\pi\varepsilon_0} \cdot \frac{(y^2-a^2)}{y^3 \cdot a\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0} \cdot \frac{a\,\cancel{y^2}\left(1-\frac{a^2}{y^2}\right)}{a\,y^{\cancel{3}} \cdot a\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=a} = \frac{q\left(\frac{a}{y}\right)\left(1-\frac{a^2}{y^2}\right)}{4\pi\varepsilon_0 \cdot a^2\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}$$

Now,

$$\sigma = -\varepsilon_0 \cdot \left.\frac{\partial \phi}{\partial x}\right|_{x=a}$$

$$\sigma = -\frac{q}{4\pi\cancel{\varepsilon_0} \cdot a^2} \cdot \cancel{\varepsilon_0}\, \frac{\left(\frac{a}{y}\right)\left(1-\frac{a^2}{y^2}\right)}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}$$

$$\sigma = -\frac{q}{4\pi a^2} \cdot \left(\frac{a}{y}\right) \frac{\left(1-\frac{a^2}{y^2}\right)}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}} \quad \text{——— (5)}$$

where, γ is the angle b/w $\underline{x}$ and $\underline{y}$. This charge density in units of $-q/4\pi a^2$ is shown plotted in fig. as a function of γ for two values of y/a.

plot b/w σ and γ:-

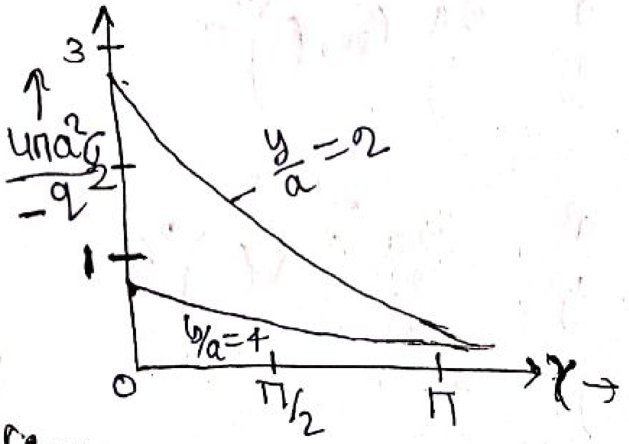

fig:- [Surface charge density σ induced on grounded sphere of radius a as a result of presence of point charge q located at a distance y away from the center of the sphere.

σ is plotted in units of $-\frac{q}{4\pi a^2}$ as a function of angular ~~function~~ position γ away from radius to the charge for $y = 2a, 4a$. Inset shows lines of force for $y = 2a$]

The Concentration of charge is in the direction of point charge q is evident specially for $y = 2a$.

It is easy to show by direct integration that total induced charge on sphere is equal to magnitude of image charge as it must be according to Gauss's law:- the force acting on charge q can be calculated in different ways; easiest way is to write down force b/w ~~image~~ charge q and image charge q', the distance b/w them:-

$$|y-y'| = y\left|1-\frac{a^2}{y^2}\right|$$

Hence, attractive force according to Coulomb's law

$$F = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3}{\left(1-\frac{a^2}{y^2}\right)^2} \quad (5)$$

it Can be obtained from this eqn,

$$F = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{q\cdot q'}{|y-y'|^2}$$

For large sepration, force follows inverse cube law $\left(\frac{1}{r^3}\right)$ but close to sphere it is propotinal to inverse square ~~law~~ of distance away from surface of sphere.